Lecture 2

<u>Theorem 1.</u> Any continuous in $[a,b]$ function f is Riemann integrable.

<u>Proof</u>. It is well known that all continuous in $[a,b]$ functions f are also uniformly continuous. Thus $\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0$ such that for meshes P:

$$\Delta x_i < \delta \Rightarrow \omega_i := M_i - m_i < \frac{\varepsilon}{b-a}.$$

Then we get:

$$\sum_{i=1}^{n} \omega_i \Delta x_i < \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Thus the sufficient and necessary condition is satisfied. $\triangleright$

Another class of integrable functions is defined by the following theorem.

Theorem 2. Monotone in $[a, b]$ functions f are Riemann integrable.

Proof Let's assume that f is an increasing function. Then

$$m_i = f(x_{i-1}), \quad M_i = f(x_i).$$

Then

$$\omega_i = M_i - m_i = f(x_i) - f(x_{i-1}).$$

and

$$\sum_{i=1}^{n} \omega_i \Delta x_i = \sum_{i=1}^{n} \underbrace{\left(f(x_i) - f(x_{i-1})\right)}_{\geq 0} \Delta x_i$$

$$< \left(\sum_{i=1}^{n} \left(f(x_i) - f(x_{i-1})\right) \right) \cdot \Delta$$

$$= \left(f(b) - f(a)\right) \Delta < \varepsilon$$

if $\Delta < \frac{\varepsilon}{f(b) - f(a)}$. $\left(\text{i.e. } U - L \xrightarrow[\Delta \to 0]{} 0 \right)$

Definition. A function f in $[a, b]$ is called *piecewise continuous* if it has in $[a, b]$ only a finite number of jump-discontinuities (first order discontinuity)

$$f(x) = \begin{cases} x^2, & 0 \le x < 1, \\ 0, & x = 1, \\ 2-(x-1)^2, & \text{for } 1 < x \le 2. \end{cases}$$

Theorem 3. A piecewise-discontinuous function f in $[a, b]$ is Riemann integrable.

Properties of definite integrals

1. Let's assume that f and g are integrable functions in $[a, b]$.

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx,$$

where $\alpha, \beta \in \mathbb{R}$ (real numbers).

Proof:

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx \overset{\text{def}}{:=} \lim_{\Delta \to 0} \sum_{i=1}^{n} (\alpha f(c_i) + \beta g(c_i)) \overbrace{}^{\Delta x_i}$$

$$= \alpha \lim_{\Delta \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i + \beta \lim_{\Delta \to 0} \sum_{i=1}^{n} g(c_i)\Delta x_i$$

$$\overset{\text{def}}{=} \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx. \quad \triangleright$$

Definition $a < b$

1. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$$

2. $$\int_a^a f(x)\,dx = 0.$$

3. For any a, b, c the following result is valid:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

<u>Proof</u>. Let's assume that $a < c < b$.

If the integral exists the limit don't depend on partition P: it can be selected such that $c \in P$, $x_{n_1} = c$.

$$\lim_{\Delta \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{\Delta \to 0} \sum_{i=1}^{n_1} f(c_i)\Delta x_i$$

$$+ \lim_{\Delta \to 0} \sum_{i=n_1+1}^{n} f(c_i)\Delta x_i = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

2) <u>$a < b < c$</u>.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$$

$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \quad \triangleright$$

4. If $f(x) \geq 0$ in $[a, b]$, then

$$\int_a^b f(x)\,dx \geq 0$$

Proof. For any mesh P:

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i \geq 0.$$ Then the limit of non-negatives numbers is also non-negative.

5. If f and g are integrable functions in $[a, b]$ and $f(x) \geq g(x)$, $\forall x \in [a, b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

Proof From (4) it follows that

$$\int_a^b \big(f(x) - g(x)\big)\,dx \geq 0,$$ but

$$\int_a^b \big(f(x) - g(x)\big)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

6. Let's assume, that

$$m = \min_{a \le x \le b} f(x), \quad M = \max_{a \le x \le b} f(x).$$

Then we get the following estimates

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \qquad (*)$$

Proof. It follows from assumptions that

$$m(b-a) = m\sum_{i=1}^{n} \Delta x_i \le \sum_{i=1}^{n} f(c_i)\,\Delta x_i \le M \sum_{i=1}^{n} \Delta x_i = M(b-a)$$

By taking a limit we get $(*)$. ▷

7. Theorem 4. If $f(x)$ is defined in $[a,b]$ and it is a continuous function, then there exist $a \le c \le b$, such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Proof. A continuous function f get its smallest and largest values in $[a,b]$ and

$$m \le f(x) \le M$$

From previous property of integrals we get

$$m \le \frac{1}{b-a} \int_a^b f(x)\,dx \le M .$$

Since f is continuous in $[a, b]$ then there exist $c \in [a, b]$

$$\frac{1}{b-a} \int_a^b f(x) = f(c) .$$

Example. Show that

$$6 \le \int_0^2 (3+x^4)\,dx \le 38 .$$

$f(x) = 3 + x^4$ is monotonic.

$m = 3 = f(0) \le f(x) \le f(2) = 19 = M \Rightarrow$ the result

Find c such that

$$\int_0^2 (3+x^4)\,dx = f(c) \cdot 2 .$$

Let f be a continuous real-valued function defined on a closed interval $[a, b]$. Thus f is integrable in $[a, b]$.

Let $F(x)$ be the function defined for all x in $[a, b]$, by

$$F(x) = \int_a^x f(t)\,dt.$$

Theorem 5. The function $F(x)$ is differentiable in open interval (a, b) and

$$F'(x) = f(x), \quad x \in (a, b).$$

Proof.

$$\Delta F = F(x + \Delta x) - F(x)$$

$$= \int_a^{x+\Delta x} f\,dt - \int_a^x f\,dt = \int_x^{x+\Delta x} f(t)\,dt$$

$$= f(c)\,\Delta x, \quad x \le c \le x+\Delta x$$

By taking the limit $\Delta x \to 0$ we get

$$\lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} \overset{def}{=} F'(x)$$

$c \to x$, when $\Delta x \to 0$ and do ~~to~~ continuity of f in (a, b)

$$\lim_{\Delta x \to 0} f(c) = f\left(\lim_{\Delta x \to 0} c\right) = f(x).$$

It is important to note, that $F(x)$ is a primitive function of f. (or antiderivative)

$$\boxed{\text{Thus for any continuous } f \text{ in } [a, b] \text{ there exists a primitive function.}}$$

<u>The Newton - Leibniz theorem.</u> If function f is continuous on $[a, b]$ and F is some primitive function of f, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

<u>Proof</u>. Since $\int_a^x f(t)\,dt$ is also a primitive function of f, a difference of it and $F(x)$ is equal to constant. Thus we can restrict to:

$$F(x) = \int_a^x f(t)\,dt + C$$

Using $x = a$ we get

$$F(a) = \int_a^a f(t)\,dt + C \Rightarrow C = F(a)$$

and

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

$$\Rightarrow \boxed{\int_a^b f(t)\,dt = F(b) - F(a).}$$

A traditional notation of the right-hand side term is the following

$$\int_a^b f(t)\,dt = F(b) - F(a) = F(x)\Big|_a^b.$$

Examples.

1. $$\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2},$$

2. $$\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

3. $$\int_0^{2\pi} (10 + \sin x + 2\cos x)\,dx$$

$$= (10x - \cos x + 2\sin x)\Big|_0^{2\pi} = \dots$$

Remark. The N-L formula is valid for any integrable function f.